Gauge theoretic invariants of surface products

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2015 CMS Winter Meeting

Overview of gauge theory

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Basic idea in gauge theory: Study X through its space of connections.

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E.g., the Donaldson invariants.

These are defined by studying the instanton moduli space.

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The *instantons* are the absolute minimizers of $\mathcal{YM}$.

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Typically take $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$ to get something new.

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However, the space $M_{\mathrm{inst}}(X)$ is, in many ways, not well-understood.

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This induces a bundle on $S \times \Sigma$ by pulling back.

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(Not very precise) Theorem

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The right-hand side is often easier to understand than the left.

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Theorem

Assume all moduli spaces are cut out transversely.
(a) [D.-McNamara, '14] There is a natural embedding

$$M_{\text{inst}}(S \times \Sigma) \hookrightarrow \left\{ u : S \to R_0(\Sigma) \mid \overline{\partial} u = 0 \right\} \tag{1}$$

whenever $\dim M_{\text{inst}}(S \times \Sigma) \leq 3$.

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whenever $\dim M_{\text{inst}}(S \times \Sigma) \leq 3$.
(b) [D., '15] If S is closed or has cylindrical ends, then (1) is a diffeomorphism that extends to a homeomorphism over the natural compactification of each space.

Discussion

The transversality assumptions can be assured using a suitable perturbation.

The proof uses the complex gauge action, and the diffeomorphism is relatively explicit.

The result extends to higher dimensional moduli spaces (bubbles/stability conditions need to be discussed, so it is more difficult to state).

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Theorem (Dostoglou-Salamon, '94)

There is an isomorphism of abelian groups

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- Take S to be S^2 with 3 punctures. Then we obtain a geometric proof of a result of Muñoz.

Theorem (Muñoz, '97)

The group isomorphism (2) is a ring isomorphism.

Future directions

Hambleton-Lee used $M_{\mathrm{inst}}(X)$ to study finite group actions on X.

Question 1: What do group actions look like on the symplectic moduli space?

Transversality in the presence of a group action is not well-understood for $M_{\mathrm{inst}}(X)$. (Hambleton-Lee use a weaker version, but do not end up with smooth moduli spaces.)

Symplectic geometers (e.g., Seidel, FOOO, Cho-Hong) have been able to define invariants on symplectic orbifolds to tackle problems in mirror symmetry.

Question 2: Can these orbifold techniques be used on our symplectic moduli space to get around equivariant transversality?

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Thank you for your attention.