Heat flows for cylindrical-end manifolds

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Two settings:

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- Yang-Mills functional for cylindrical-end 4-manifolds

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- Energy functional for cylindrical-end surfaces into a symplectic manifold

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More specifically, the gradient flows provide a weaker alternative to the implicit function theorem. (More later.)

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Assume all ASD connections are regular.

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$$\mathcal{YM} : \mathcal{A}(Z; a) \to \mathbb{R}, \qquad A \mapsto \frac{1}{2}\|F_A\|^2_{L^2}.$$

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$$\partial_\tau A(\tau) = -d^*_{A(\tau)} F_{A(\tau)}, \qquad A(0) = A_0$$

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This is the *Yang-Mills heat flow starting at* A_0.

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$$\|F^+_{A_0}\|_{L^2} < \eta,$$

then there exists a unique solution $A : [0, \infty) \to \mathcal{A}(Z; a)$ *to the Yang-Mills heat flow starting at* A_0,

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then there exists a unique solution $A : [0, \infty) \to \mathcal{A}(Z; a)$ *to the Yang-Mills heat flow starting at* A_0, *and there is an ASD connection* A_∞ *so that*

$$\lim_{\tau \to \infty} A(\tau) = A_\infty$$

where the convergence is exponential and in $\mathcal{C}^\infty(Z)$.

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This extends work of Struwe '94 and Schlatter '97, working in the compact setting.

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Work of Waldron '16 suggests this is not even necessary!

Future directions

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- Can this help to understand Floer moduli spaces in the absence of perturbations?

Thank you for your attention!