Constructions Regarding Integration in the Plane and the Rotation of Segments

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August 14, 2006

Abstract

In 1919 A. S. Besicovitch's interest in plane integration led him to construct an integrable function defined in the plane that is not integrable as an iterated integral for any pair of mutually orthogonal directions. Later, in 1928 he noticed that this construction could be modified in such a manner that would suffice as a counterexample to S. Kakeya's conjecture that the smallest area required to rotate a unit line segment continuously in a plane is $\pi/8$, by constructing a set that allowed for this continual rotation in arbitrarily small area. In this paper we reconstruct both Besicovitch's integration example and his counterexample to Kakeya's conjecture.

This paper will investigate two questions, each posed in the first part of the twentieth century. The first, asked by A. S. Besicovitch, is a problem in Riemann integration and reads

> I) 'Given a function of two variables, Riemann integrable on a plane domain, does there always exist a pair of mutually perpendicular directions such that repeated simple integration along these two directions exists and gives the value of the integral over the domain?' [3]

The second was inspired by S. Kakeya in 1917. Kakeya was considering the question

> II) Given a unit line segment in a plane, what is the figure having smallest area in which the segment is able to completely rotate within the figure?

For example a circle having diameter 1 would suffice by simply allowing the line segment to rotate about its fixed center. However, the rotation of a segment is also possible within an equilateral triangle of unit height. Kakeya conjectured that if the figure were required to be convex then the figure of smallest area that allowed for the rotation of a unit segment would in fact be an equilateral triangle with unit height. In 1921 J. P$\acute{a}$l proved this to be correct [5]. Kakeya also noted that if the convexity restriction was not imposed then there exists a figure with area smaller even than the equilateral triangle, namely the three-cusped hypocycloid with area $\pi/8$ pictured below in Figure 1 [5]. Convexity is however a very strong assumption and lifting this requirement allows for many other possible figures in which a unit line segment can be completely rotated, thus establishing the conjecture in II), which became known as the Kakeya Conjecture.

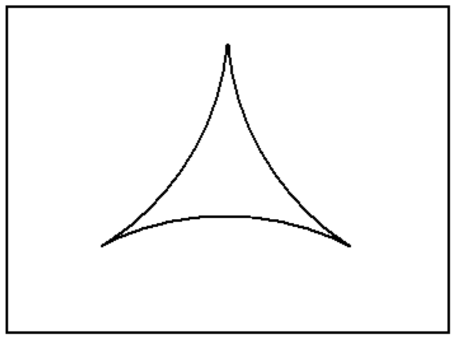


Figure 1: The hypocycloid with an area of $\pi/8$ allows for the continuous rotation of a unit segment.

In the year 1919 Besicovitch provided a solution to his integration question in I). He was able to show that the answer to I) is a resounding *No* [2], thus reinforcing that plane integration and iterated integration are inherently (although subtly) different objects. To do this, Besicovitch constructed a set of Jordan measure zero that contained a unit line segment in each direction. He then manipulated this set and cleverly defined a function that is integrable in the plane but is not as an iterated integral for any pair of mutually orthogonal directions. It was using this set that Besicovitch was later able to construct a set that would supply an answer to II), that is that there exists such a set with arbitrarily small area! Our objective here is to reproduce the answers to I) and II) which we will do by first constructing a set of arbitrarily small area that contains a unit line segment in each direction as Besicovitch originally did in 1919. However, the downside of this is that the resulting set was arbitrarily large, multiply-connected and rather difficult to construct. However, in past years there have been vast improvements on Besicovitch's original construction. For instance, Cunningham constructs a simply-connected Kakeya set with arbitrarily small area that is contained in a ball of diameter 1 [4]! Due to its simplicity however, we will follow a construction by Falconer [5] to demonstrate the solution to II) that there exists such a figure with arbitrarily small area. Upon creating this set we will then diverge from Falconer's construction and follow that of Besicovitch in [2] to obtain an answer to the question in I). It is interesting to note that Besicovitch also constructed a function that is integrable as an iterated integral for every pair of orthogonal directions but is not integrable in the plane [2], which further strengthens the notion plane and iterated integration are not innately the same. We will examine Besicovitch's construction of this function.

For ease of reference we will adopt the term *Kakeya Set* to denote a set in a plane in which a unit line segment can be turned through 180° by a continuous movement, where by continuous movement of the segment we mean that the endpoints of the segment follow paths that are images of continuous functions in one parameter [1]. Furthermore, recall that when dealing with compact sets the notions of Jordan measure and Lebesgue measure are equivalent. Hence, we may unambiguously refer to the Jordan/Lebesgue measure of a compact set S as its *measure* or its *area* and denote it by $A(S)$. Lastly, all integrals considered are Riemann integrals.

1 Preliminaries.

The following will be needed to construct solutions to both I) and II); these results are slight modifications of those made by Falconer in [5]. The idea in this section is to take a triangle, slice it up into a sufficiently large amount of smaller triangles, then to translate these smaller triangles in such a manner that the resulting figure has as small of an area as we please. We will do this so that if the original triangle allows for the complete revolution of a unit segment then the resulting figure will too. This process however will take a number of steps and this first lemma will provide us with some useful estimates on these 'slicing up' and 'translating' operations.

Lemma 1. Let T_1 and T_2 be adjacent triangles with bases on a line L, each having base length b and height h. Take $\alpha \in (\frac{1}{2}, 1)$ and consider the new figure $\widetilde{T_2}$, which is the translation of T_2 a distance $2(1-\alpha)b$ along L in the direction of T_1. Denote by S the union of T_1 and $\widetilde{T_2}$. Then:

i) The figure $S = T_1 \cup \widetilde{T_2}$ consists of a triangle, T, having base on line L, and two smaller triangles, t_1 and t_2, which we will call *auxiliary triangles*. The triangle T is similar to $T_1 \cup T_2$ and is positioned in a similar manner.

ii) $A(T) = \alpha^2 A(T_1 \cup T_2)$.

iii) The difference in area between $T_1 \cup T_2$ and S is given by

$$A(T_1 \cup T_2) - A(S) = A(T_1 \cup T_2)(1-\alpha)(3\alpha - 1)$$

where A is plane area.

Proof. Consider Figures 2 and 3; we will prove i-iii each in turn.

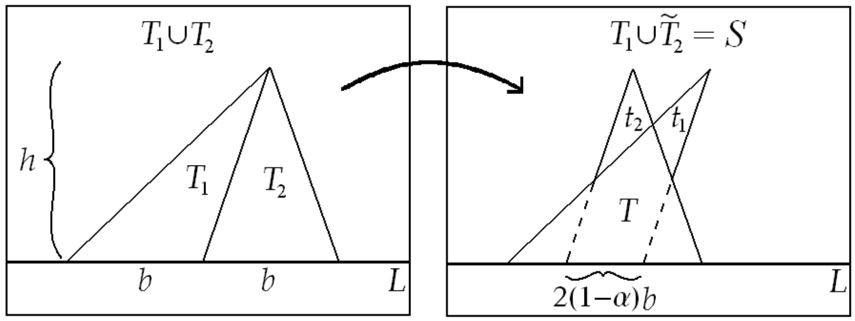


Figure 2:

i) Clearly $m\angle 1 = m\angle 1'$ and $m\angle 2 = m\angle 2'$, hence $T \approx T_1 \cup T_2$ (the symbol $\approx$ is used here to denote similarity). The latter assertion in i) regarding the positioning of the triangles is obvious.

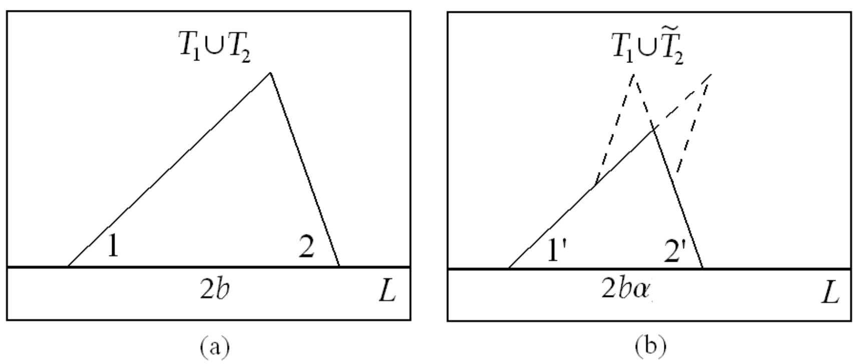


Figure 3: The triangle in (a) is similar to the triangle in (b) that has a base on L.

ii) We see that T has a base of length $2b - 2(1-\alpha)b = 2\alpha b$; whereas $T_1 \cup T_2$ has base of length $2b$. So the ratio of similitude between T and $T_1 \cup T_2$ is α, hence $A(T) = \alpha^2 A(T_1 \cup T_2)$.

iii) Let l be the line parallel to L and passing through the common vertex shared by t_1 and t_2, as in Figure 4. Consider first the triangle t_1. l divides t_1 into two triangles, call them $t_{1,1}$ and $t_{1,2}$ where $t_{1,2}$ is between l and L. Let x be the length of line segment $l \cap t_1$. x depends linearly on α, so treating x as a function in α and noting that $x(\alpha = 1) = 0$ and $x(\alpha = \frac{1}{2}) = \frac{b}{2}$ we see that $x = b(1-\alpha)$. Notice that $t_{1,1} \approx T_1$ and $t_{1,2} \approx T_2$, each having a ratio of similitude $\frac{x}{b} = 1 - \alpha$. Hence $A(t_{1,1} \cup t_{1,2}) = (1-\alpha)^2 A(T_1 \cup T_2)$.

For t_2 let $y = t_2 \cap l$. Just as above we get $y = b(1-\alpha)$, so $x = y$ and we can conclude that t_1 is congruent to t_2. Now the total area of S is given by

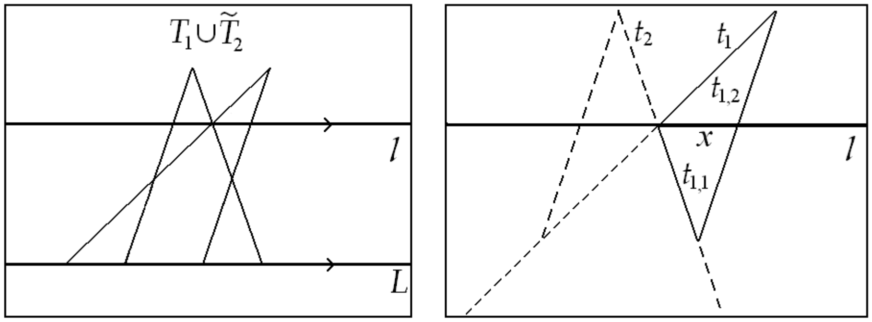


Figure 4: Lines L and l are parallel. Notice that $t_{1,2} \approx T_1$ and $t_{1,1} \approx T_2$ (see Figure 2)

$$\begin{aligned} A(S) &= A(T) + A(t_1) + A(t_2) \\ &= \alpha^2 A(T_1 \cup T_2) + 2(1-\alpha)^2 A(T_1 \cup T_2) \end{aligned}$$

and, after a little algebra, we arrive at

$$A(T_1 \cup T_2) - A(S) = A(T_1 \cup T_2)(1-\alpha)(3\alpha - 1).$$

Q.E.D.

Now we will use the estimates from the previous lemma to perform the aforementioned 'slicing up' and 'translating' of a triangle to form a new figure with arbitrarily small area.

Lemma 2. Consider a triangle, T, having base on a line L. Partition the base of T into 2^k congruent segments and join each of the endpoints of these segments to the vertex opposite the base, forming 2^k triangles, $T_1, \ldots, T_{2^k}$. Then for every $\epsilon > 0$ there exists K so that when $k > K$ it is possible to translate these 2^k triangles along L to form a new figure, S, in such a manner that $A(S) < \epsilon$.

Remark. The translation of each triangle T_i as described in Lemma 2 is applied to the triangle *and* its boundary. That is, the image of each T_i under the translation is a closed figure. Thus the resultant figure, S, is compact. Notice that since some boundaries are shared by two triangles we have 'added in' $2^k - 1$ more line segments. This however does not effect area since there are only finitely many of these new segments.

Proof. Fix some $\epsilon > 0$. We would like to employ Lemma 1, however in doing so we must specify a value for $\alpha \in (\frac{1}{2}, 1)$. To attain the desired result, the value α must be chosen so that

$$\frac{1 + (1 - \frac{\epsilon}{A(T)})^{1/2}}{3 - (1 - \frac{\epsilon}{A(T)})^{1/2}} < \alpha < 1. \tag{1}$$

In order to justify that it is even possible to pick an α satisfying (1) we must show that the quantity $\frac{1+(1-\frac{\epsilon}{A(T)})^{1/2}}{3-(1-\frac{\epsilon}{A(T)})^{1/2}}$ in (1) is indeed less than 1. By assumption

$$0 < \frac{\epsilon}{A(T)},$$

$$1 > 1 - \frac{\epsilon}{A(T)}.$$

Taking the square root of both sides and multiplying by 2 gives

$$2 > 2\left(1 - \frac{\epsilon}{A(T)}\right)^{1/2},$$

hence

$$3 - \left(1 - \frac{\epsilon}{A(T)}\right)^{1/2} > 1 + \left(1 - \frac{\epsilon}{A(T)}\right)^{1/2}.$$

Dividing both sides by $3 - \left(1 - \frac{\epsilon}{A(T)}\right)^{1/2}$ yields the desired result.

Our other preliminary is to specify a value for K. Although at the moment the motivation may seem ambiguous, let

$$K > log_{\alpha^2}\left[1 - (1 - \frac{\epsilon}{A(T)})^{1/2}\right], \tag{2}$$

and choose $k > K$. With this value of k, construct the 2^k triangles in the manner indicated in the hypotheses. By the following repeated application of the previous lemma we will construct the desired set, S.

Step 1. Consider two consecutive triangles T_{2i-1} and T_{2i}. Translate T_{2i} along L in the direction of T_{2i-1}, a distance of $2(1-\alpha)b$, where b is the length of the base (the base lying on L) of each T_j. This translation forms a new figure, which we will call S_i^1. By Lemma 1 the figure S_i^1 is the union of two auxiliary triangles and a triangle T_i^1 that is similar to $T_{2i-1} \cup T_{2i}$ and positioned in a similar manner; furthermore $A(T_i^1) = \alpha^2 A(T_{2i} \cup T_{2i-1})$. The reduction in area in replacing

$T_{2i-1} \cup T_{2i}$ with the new figure S_i^1 is $A(T_{2i} \cup T_{2i-1})(1-\alpha)(3\alpha-1)$. Repeating for each $1 \leq i \leq 2^{k-1}$ yields a collection, $\{S_i^1\}$, of new figures. Figures 5 and 6 illustrate the collection $\{S_i^1\}$ for $k = 3$.

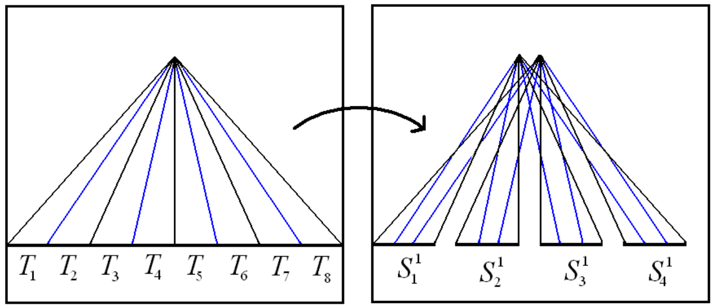


Figure 5: Translating the T_i to form the S_i^1 when $k = 3$.

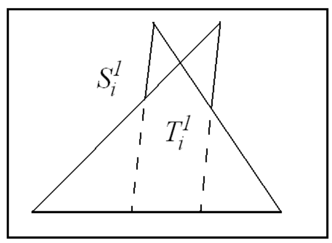


Figure 6:

Step 2. We will now perform a similar operation with consecutive S_i^1. Let $1 \leq i \leq 2^{k-2}$ and translate S_{2i}^1 along L in the direction of S_{2i-1}^1. Call this new figure S_i^2. See Figure 7. Restricting our attention to the action of T_{2i-1}^1 relative to T_{2i}^1 in this translation we notice that one side of T_{2i-1}^1 is parallel and congruent to the opposite side of T_{2i}^1. Hence, Lemma 1 allows us to perform this translation so that S_i^2 contains some triangle T_i^2 where $A(T_i^2) = \alpha^2 \left(A(T_{2i-1}^1) + A(T_{2i}^1) \right)$ and the reduction of area achieved by replacing $T_{2i-1}^1 \cup T_{2i}^1$ by S_i^2 is at least

$$(1-\alpha)(3\alpha-1)\left(A(T_{2i-1}^1) + A(T_{2i}^1) \right)$$

$$= (1-\alpha)(3\alpha-1)\alpha^2 A(T_{4i-3} \cup T_{4i-2} \cup T_{4i-1} \cup T_{4i}).$$

Step 3. Inductively, let $j \leq k$ and suppose we have j collections of figures $\{S_i^m\}_{i=1}^{2^{k-m}}$ for each $m \leq j$, satisfying the following conditions:

i) each S_i^m lies on L,

ii) each S_i^m contains some triangle T_i^m that also lies on L,

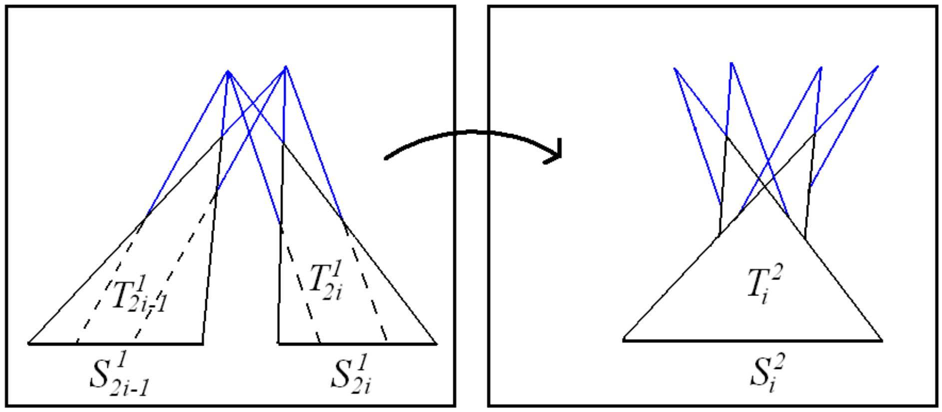


Figure 7: Here S^1_{2i+1} is being translated to overlap S^1_{2i}, thus forming S^2_i.

iii) for a fixed m the T^m_i are disjoint ,

iv) for a fixed m consecutive T^m_i have one pair of congruent, parallel sides,

v) $A(T^{m-1}_{2i-1} \cup T^{m-1}_{2i}) - A(S^m_i) \geq (1-\alpha)(3\alpha - 1)\left(A(T^{m-1}_{2i-1}) + A(T^{m-1}_{2i})\right)$,

vi) $A(T^m_i) = \alpha^2 A(T^{m-1}_{2i-1} \cup T^{m-1}_{2i})$.

With $1 \leq i \leq 2^{k-j}$ translate S^j_{2i} along L to overlap S^j_{2i-1} obtaining a new figure S^{j+1}_i. Lemma 1 tells us that this can be done so that each of the six conditions above are satisfied for $m = j+1$, thus completing the inductive step.

Notice that condition ii) provides us with countable additivity, hence by repeated application of condition v) above we have

$$A(S^k_1) \leq \left[A(T^{k-1}_1) + A(T^{k-1}_2)\right] - (1-\alpha)(3\alpha-1)\left[A(T^{k-1}_1) + A(T^{k-1}_2)\right]$$

$$\leq \left[A(S^{k-1}_1) + A(S^{k-1}_2)\right] - (1-\alpha)(3\alpha-1)\left[A(T^{k-1}_1) + A(T^{k-1}_2)\right]$$

$$\leq \left[A(T^{k-2}_1) + \ldots + A(T^{k-2}_4)\right] - (1-\alpha)(3\alpha-1)\left[A(T^{k-2}_1) + \ldots + A(T^{k-2}_4)\right] -$$
$$(1-\alpha)(3\alpha-1)\left[A(T^{k-1}_1) + A(T^{k-1}_2)\right]$$

$$\vdots \qquad \qquad \vdots$$

$$\leq \left[A(T_1) + \ldots + A(T_{2^k})\right] - (1-\alpha)(3\alpha-1)\left[A(T_1) + \ldots + A(T_{k^2})\right] -$$

$$(1-\alpha)(3\alpha-1)\left[A(T_1^1)+\ldots+A(T_{2^{2(k-1)}}^{k-1})\right]-\ldots-(1-\alpha)(3\alpha-1)\left[A(T_1^{k-1})+A(T_2^{k-1})\right].$$

Now use condition vi) to get

$$=A(T)-(1-\alpha)(3\alpha-1)(1+\alpha^2+\ldots+\alpha^{2(k-1)})A(T)$$

$$=\left(1-\frac{(3\alpha-1)(1-\alpha^{2k})}{1+\alpha}\right)A(T). \tag{3}$$

This is our sought after estimate, so we will take S_1^k to be our set S. We will now manipulate equations (1) and (2) to complete the proof. Solving (1) for the quantity $(1-\frac{\epsilon}{A(T)})^{1/2}$ gives us

$$\left(1-\frac{\epsilon}{A(T)}\right)^{1/2}<\frac{3\alpha-1}{1+\alpha},$$

and by our choice of k in (2) we get

$$1-\alpha^{2k}<\left(1-\frac{\epsilon}{A(T)}\right)^{1/2}.$$[1]

So by putting these together with (3) we arrive at

$$A(S)<\epsilon.$$

Q.E.D.

Remark. Notice that by fixing the position of the first subtriangle, T_1, and performing the above operations with respect to T_1, each T_i will have moved no more than b along L. See Figure 8. We will use this idea in the theorem below.

Theorem 1. With the same notation as Lemma 2, given some open set $V\supset T$ and $\epsilon>0$, the construction of S can be done so that $S\subset V$ and $A(S)<\epsilon$.

Proof. Let $V\supset T$ be an open set and fix $\epsilon>0$. Notice that the only part of V that we are interested in is that which is very close to the triangle T, so since T is bounded we may as

[1]Since $\alpha<1$ we have that $f(x)=\alpha^{2x}$ is a *decreasing* function, so the inequality in (2) is reversed when both sides are raised to a power of α^2.

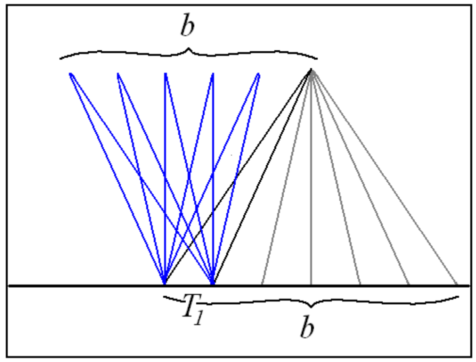


Figure 8: The blue triangles are the images of the gray triangles after translating. Notice that all of the translating is done with respect to T_1 and each triangle has been translated no more than b.

well assume that V is bounded as well. Later on this will allow us the luxury of using compactness. We want to be able translate the points in T to make its area very small, but we cannot translate the points very far since we need to stay within V. So we first find an upper bound, δ, exhibiting the property that if any point of T is translated a distance less than δ in any direction then the point will still be in V.

To find δ we define a function $\phi : T \to \mathbf{R}$ so that

$$\phi(p) = min\left\{|p - x| : x \in \partial V\right\},$$

where ∂V denotes the boundary of V. Note that by the extreme value theorem the distance function achieves a minimum on ∂V, so ϕ is well defined. Furthermore, a calculation shows that ϕ is continuous, so by the compactness of T and another application of the extreme value theorem, ϕ achieves a minimum. This minimum will be our δ. Now choose $n > b/\delta$, where b is the length of the base of T. Divide up the base of T into n subtriangles, $T_1, \ldots, T_n$. We will now apply Lemma 2 to these subtriangles (observe that we are *not* applying Lemma 2 to the larger triangle T). The case for $n = 4$ is shown in Figure 9.

Consider some T_i. By Lemma 2 we can divide T_i into smaller subsubtriangles, so that, upon translation, we obtain a figure S_i where $A(S_i) < \epsilon/n$. Furthermore, by the remark following Lemma 2 this can be done so that each subsubtriangle has moved a distance less than the length of the base of T_i. Or equivalently, the elements of S_i have been moved no more than $b/n < \delta$, so the elements of S_i are still contained in V. Repeating this for each of the T_i one obtains a new collection of figures, $\{S_i\}$. Let $S = \cup_{i=1}^{n} S_i$. We thus have

$$A(S) = A(\cup_{i=1}^{n} S_i) \le \sum_{i=1}^{n} A(S_i) < \epsilon,$$

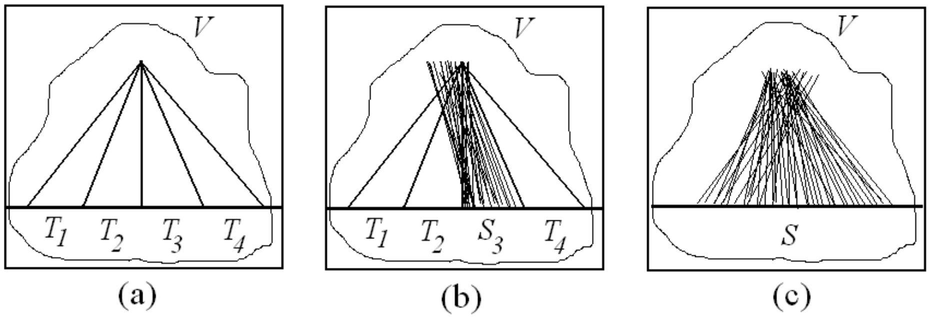


Figure 9: Illustrated above is the case for $n = 4$. (b) shows Lemma 2 applied to T_3. (c) shows Lemma 2 applied to all of the subtriangles.

and furthermore, in obtaining S from T the elements have been translated no more than δ, so $S \subset V$ as desired.

Q.E.D.

Theorem 2. There exists a bounded set with zero area containing a unit line segment in every direction from 0° to 90°. Furthermore, given a line L this set can be constructed so that each segment has an endpoint on L and all of the segments lie on the same side of L.

Proof. Let S_1 be an isosceles right triangle with unit height and having its longest base on a line L. S_1 is measurable so there exists an open cover of S_1 with measure as close to $A(S_1)$ as we please. So let $V_1 \supset S_1$ be an open set such that $A(\overline{V_1}) \leq 2A(S_1)$, where $\overline{V_1}$ denotes closure. Notice that by taking an intersection if necessary, the set V_1 is easily contained in a ball of radius 2, see Figure 10. Now applying Theorem 1 to S_1 we obtain a new closed figure, $S_2 \subset V_1$, that is the finite union of triangles, each having base on line L and with $A(S_2) \leq 2^{-2}$.

Since S_2 is measurable there exists an open set, V_2, containing S_2 and with measure as close to $A(S_2)$ as we please. So choose V_2 so that $A(\overline{V_2}) \leq 2A(S_2)$. V_1 contains S_2 and is open so, by taking an intersection if necessary, this can be done so that $S_2 \subset V_2 \subset V_1$. By applying Theorem 1 again to each of the triangles that make up S_2 we obtain a new set, $S_3 \subset V_2$, with $A(S_3) \leq 2^{-3}$.

Repeating this process iteratively we obtain two collections of sets, $\{V_i\}$ and $\{S_i\}$, so that each set satisfies the following properties:

i) each V_i is open,

ii) $S_i \subset V_i$,

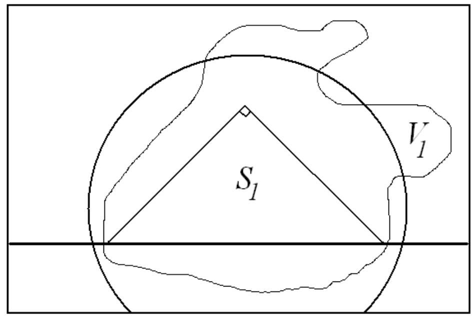


Figure 10: Taking an intersection of V_1 and a circle of radius 2 that contains S_1 ensures that V_1 is bounded.

iii) $V_i \subset V_{i-1}$,

iv) $A(\overline{V_i}) \leq 2A(S_i) \leq 2^{-i+1}$,

v) each S_i is the finite union of triangles having unit height and base on L; hence each S_i has a unit line segment in each direction between 0° and 90° that lies on one side of L and has an endpoint on L.

Set

$$S = \bigcap_{i=1}^{\infty} \overline{V_i}.$$

Notice that S is closed and $S \subset V_1$, the latter of which is bounded so S is compact. Hence the notions of Jordan measure and Lebesgue measure are equivalent. Property iv) above implies that S has zero Lebesgue measure and hence $A(S) = 0$. It remains to show that S contains a line segment in every direction between 45° and 135° (measured with respect to L) as this statement, along with property v), implies that each line segment lies on one side of L and contains an endpoint on L. So let θ be an angle in the range $[45, 135]$. By property v) for each i there exists some line segment $M_i \subset S_i$ making an angle θ with L. Property ii) then tells us that $M_i \subset \overline{V_i}$. Let x_i be the x-coordinate of the endpoint of M_i that lies on L, see Figure 11. It then follows from property iii) that $\{x_i\} \subset \overline{V_1}$. Compactness of $\overline{V_1}$ allows us to pass to a convergent subsequence of $\{x_i\}$, so let x be the limit of this subsequence and let M be the line segment corresponding to this value of x, that is the line segment whose endpoint that lies on L has x-coordinate x. Notice that if $i \geq j$, then $M_i \subset \overline{V_j}$ by property 3. So since the M_i become arbitrarily close to M and as each $\overline{V_i}$ is closed we get that $M \subset \overline{V_i}$ for each i. Hence $M \subset S$ as desired.

Q.E.D.

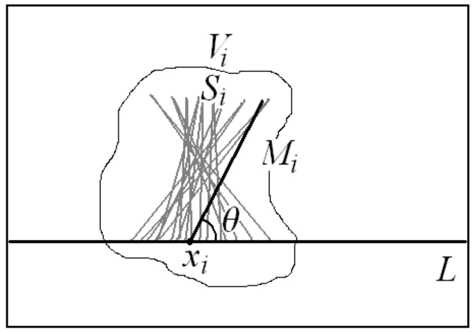


Figure 11: The bolded line segment is M_i and intersects the x-axis at x_0, forming an angle θ. S_i is illustrated by the gray line segments and V_i is the open set containing S_i.

2 Rotating Line Segments.

We will use the following lemma to construct a Kakeya set with arbitrarily small area.

Lemma 3. Let L_1 and L_2 be parallel lines. For any $\epsilon > 0$ there exists a compact set E with area less than ϵ in which a unit line segment can be moved continuously from L_1 to L_2.

Proof. Let ω be the distance between L_1 and L_2 and let x_1 be any point on L_1. Take x_2 to be a point on L_2 a distance D away from the projection of x_1 onto L_2, where

$$D > \frac{\omega}{tan\ \pi\epsilon}.$$

Denote by M the linesegment connecting x_1 with x_2. The continuous movement we are looking for will take the unit line segment from line L_1, rotate it to M about the point x_1. Then slide the segment to x_2 and rotate around x_2 until it lies on L_2. To allow for this we must include the two congruent sectors, S_1, S_2, of radius 1 that lie between M and L_1, L_2, respectively. The angle between M and each L_i is $tan^{-1}\frac{\omega}{D} < \epsilon\pi$, so the area of each sector is less than $\epsilon/2$. By taking E to be the union of M with S_1 and S_2 we attain our desired set.

Q.E.D.

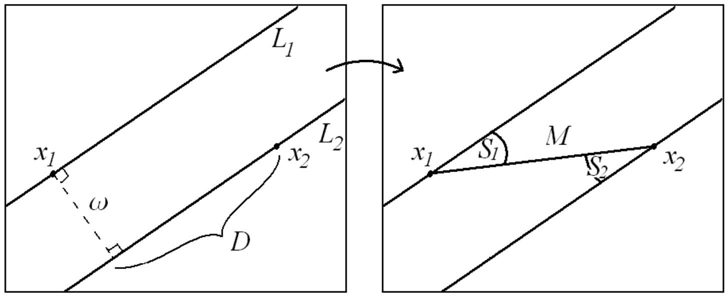


Figure 12:

Theorem 3. Given $\epsilon > 0$ there exists a Kakeya set with Jordan measure less than ϵ.

Proof. Take an isosceles right triangle having unit height and longest base on a line L. Lemma 2 enables us to cut up this triangle into n smaller triangles, $\{T_i\}_1^n$, of unit height each having base on L, where $A(\cup_1^n T_i) < \epsilon/6$. So in each T_i a unit line segment can be rotated from one side to the other in a continuous fashion. Notice that T_i and T_{i+1} have two sides that are parallel. By an application of Lemma 3 we can move the segment from T_i to T_{i+1} in an area that is less than $\epsilon/6n$. So as the segment moves from each T_i to the next and through each T_i, and hence through $90°$, it sweeps out an area that is less than $\epsilon/3$.

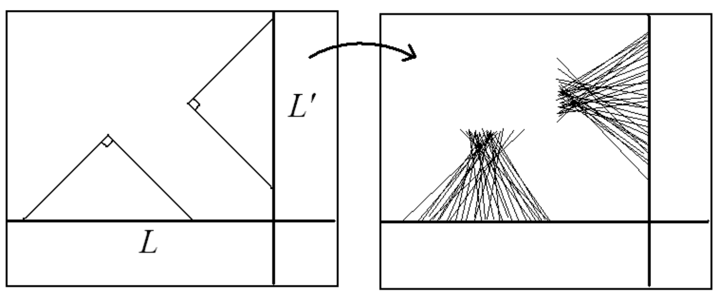


Figure 13:

Now take L' to be a line perpendicular to L and construct a copy of the above set, but this time with respect to L'. Notice that these sets contain one pair of parallel line segments, so by Lemma 3 a unit line segment can be moved continuously from one set to the other in an area that is less than $\epsilon/3$. Thus taking the union of these sets we obtain a Kakeya set with area less than ϵ.

Q.E.D.

3 Integration.

Before even learning of Kakeya's conjecture Besicovitch was studying Riemann integration in the plane and the conditions under which plane integration is equivalent to iterated integration, as in the equality observed in Fubini's theorem (see Appendix). In [2] he demonstrated the limitations of this equivalence by constructing functions and sets where iterated integrals exist but the corresponding plane integrals do not and where integrals exist but the iterated integrals do not. We will look into both of these examples below and will follow Besicovitch's original construction.

Theorem 4. There exists a function $f : \mathbf{R^2} \to \mathbf{R}$ and a set $S \subset \mathbf{R^2}$ so that the Riemann integral

$$\int\int_S f\, dA$$

exists, but the expression

$$\int_\alpha^\beta \int_{\eta_1}^{\eta_2} f\, d\xi\, d\eta$$

does not exist as a Riemann integral for any pair of orthogonal directions η, ξ, where S lies between curves η_1 and η_2, which ranges from α to β in the ξ direction.

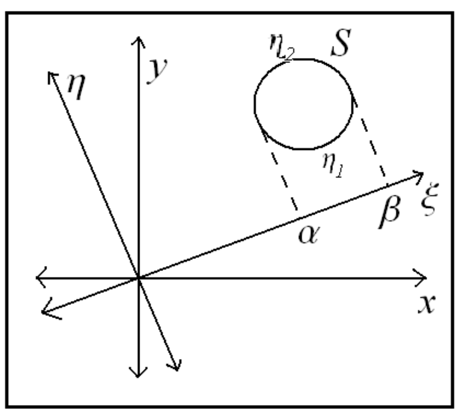


Figure 14: The set S and an arbitrary pair of orthogonal axes, η and xi.

Proof. Let B be a set as in Theorem 2, taking L to be the x-axis where the line segments are positioned in the first quadrant. Let $r_1, r_2, \ldots$ be an enumeration of the rationals in the range $(0, 1)$ and consider the set $\{L_i\}$ of lines where L_i is the line $y = 2^{-i}$. Denote by B_i the sections of line segments of B that lie between L_i and L_{i+1}. For each i, translate B_i a distance r_i in the direction of the positive x-axis. Thus we have a new set which we call S_1. See Figure 15.

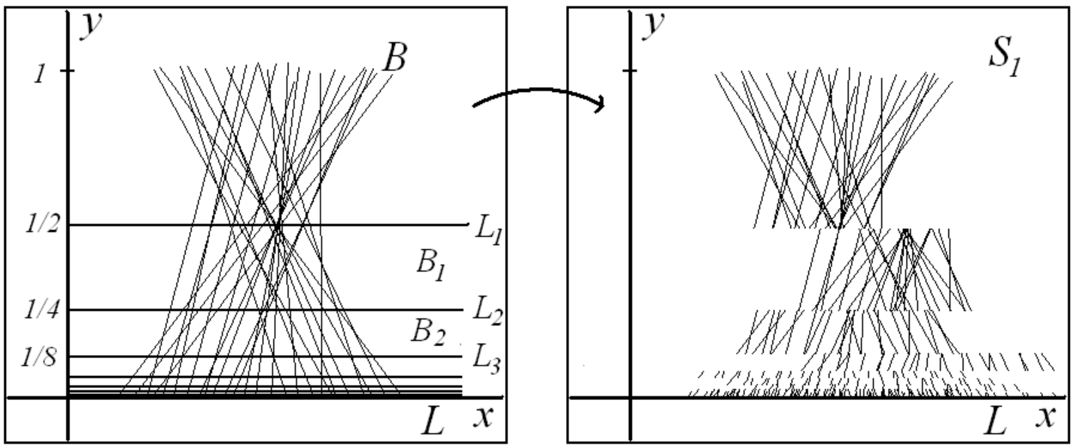


Figure 15: Illustrated above is the translations of the B_i.

Our claim now is that S_1 has zero area (Jordan measure). The difficulty comes in that S_1 is the result of an infinite number of translations and Jordan measure does not handle infinities very well. However, these infinite translations all occur very close to the x-axis, so by placing a sufficiently wide rectangle on the x-axis we can negate this problem. We will do this as follows: choose any positive ϵ and recall that B is bounded, so S_1 must also be bounded since it was constructed from B by translating distances no greater than 1. So let ψ be the length of an interval that contains the projection of B onto the x-axis. This will be the length of the base of our rectangle and we will take $\epsilon/2\psi$ to be its height. There is no harm in insisting that this rectangle be closed, so we will require it here as it will benefit us later. Position the base of this rectangle on the x-axis so that is covers the lower portion of S_1 as in Figure 16; this is possible because we have chosen the width to be great enough. The remaining uncovered portion of S_1 consists of a finite number of translates of B and hence there exists a finite closed cover of this portion of S_1 with area less than $\epsilon/2$ and we have the freedom to insist that this cover consist solely of rectangles. Thus we have a finite cover of S_1 that consists of closed rectangles with a total area that is less than ϵ.

Now let S_2 be the image of S_1 after a 90° rotation. Observe that $A(S_1 \cup S_2) \leq A(S_1) + A(S_2) = 0$. We are now ready to define the function f as follows: (i) if $P \in S_1$ has a rational x-coordinate then $f(P) = 1$, (ii) if $P \in S_2$ has a rational y-coordinate then $f(P) = 1$, (iii) $f(P) = 0$ for all

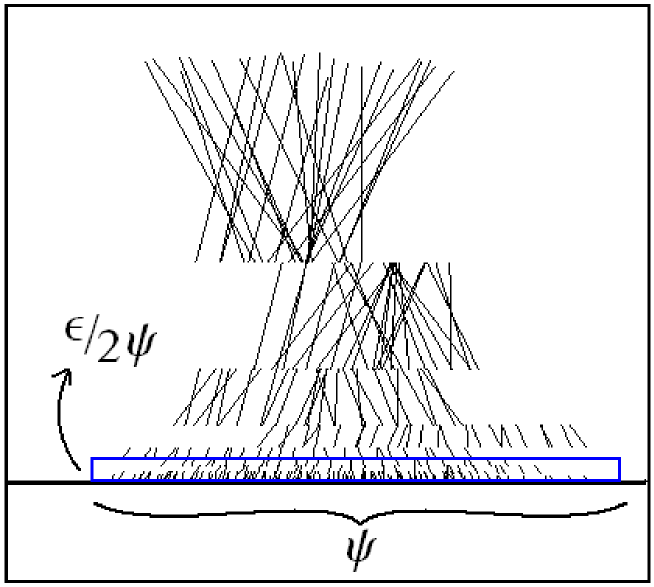


Figure 16: The rectangle in blue has area $\epsilon/2$.

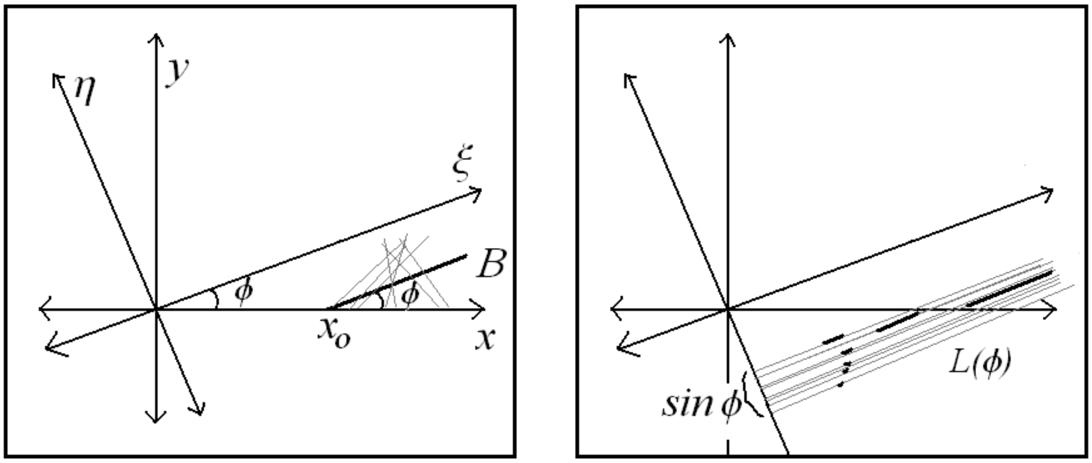
other P. This makes ensures that f is not integrable as a function of a single variable, ξ or ν, on any of the line segments of S_1 or S_2, respectively. Let S be a ball that contains $S_1 \cup S_2$ and its finite cover of closed rectangles that was constructed above. The complement of the cover of $S_1 \cup S_2$ is open and since f is constantly zero on this cover it follows that f is continuous there. Thus f is discontinuous at most on the cover of rectangles, which can be made to have as small of an area as we please, so the integral

$$\int\int_S f\, dA$$

exists and equals zero.

Let η and ξ be an arbitrary pair of orthogonal axes, where η is the axis that makes the smaller angle with the x-axis, if they both meet the x-axis at the same angle then just choose one of the axes to be ξ. Call this angle formed ξ. Consider the line segment of B that makes an angle ϕ with the x-axis and let x_0 be the x-coordinate of the intersection of this line segment with the x-axis. Now let $L(\phi)$ denote the set of parallel lines that contain the line segments of S_1 which make an angle ϕ with the x-axis, see Figure 17. Notice that $L(\phi)$ can be described by $\{(x, y) : y = tan(\phi)(x - x_0 - r), \forall r \in \mathbf{Q} \cap (0, 1)\}$. By the density of the rationals it follows that the intersection of $L(\phi)$ with the x-axis is dense in the interval $(x_0, x_0 + 1)$. If we now consider the intersection of $L(\phi)$ with the ν-axis we obtain a set that is dense in an interval of length $sin\ \phi \geq \sqrt{2}/2$.

Recall that f is not integrable over any of the line segments of S_1 or S_2, hence the integral



(a) (b)

Figure 17: The dark line segment in (a) is the segment of B that is parallel to the ξ-axis. The gray lines in (b) are the lines of $L(\phi)$ that each contain a line segment from S_1.

$$\int_{\eta_1}^{\eta_2} f \, d\xi$$

does not make sense over any of the lines in $L(\phi)$ when considered as a Riemann integral. So when viewed as a function in η the expression $\int_{\eta_1}^{\eta_2} f(\xi, \eta) \, d\xi$ is not defined on a dense set of length $sin \ \phi$. Hence, in considering the expression

$$\int_{\alpha}^{\beta} \int_{\eta_1}^{\eta_2} f \, d\xi \, d\eta$$

we conclude that it does not exist as a Riemann integral. By the same argument it follows that

$$\int_{\delta}^{\gamma} \int_{\xi_1}^{\xi_2} f \, d\eta \, d\xi$$

does not exist, thus completing the proof.

Q.E.D.

The following theorem does not use any of the machinery developed previously in its proof, however it is included here because its results are so similar to that of the previous theorem. As a note, the following also follows in the manner of Besicovitch in his 1919 paper.

Theorem 5. There exists a function $f : \mathbf{R^2} \to \mathbf{R}$ and a set $S \subset \mathbf{R^2}$ so that the iterated integral

$$\int_{\alpha}^{\beta} \int_{\eta_1}^{\eta_2} f \, d\xi \, d\eta$$

exists for every pair of orthogonal directions η, ξ, but the expression

$$\int \int_S f \, dA$$

does not exist, where S is bounded in the η direction by the curves η_1 and η_2 and ξ ranges from α to β.

Proof. Consider the square with vertices $A(0,1)$, $B(1,1)$, $C(1,0)$ and $D(0,0)$. Let $r_1, r_2, \ldots$ be an enumeration of the rationals in $(0,1)$. For each i denote by L_i the translation of line segment DA in the direction of the positive x-axis a distance of r_i. Divide each L_i into i equal pieces and designate the points of division by $a_{i,1}, a_{i,2} \ldots a_{i,i-1}$.

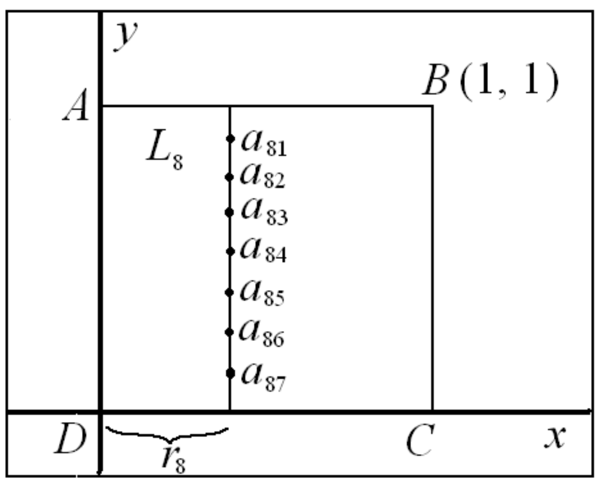


Figure 18: Shown here is the case where $i = 8$.

We are now going to construct the set of points

$$b_{2,1}$$

$$b_{3,1}, b_{3,2}$$

$$\vdots \qquad\qquad \vdots$$

$$b_{i,1}, b_{i,2}, \ldots, b_{i,i-1}$$
$$\vdots \qquad\qquad\qquad \vdots$$

that satisfy the properties

i) the distance from $b_{i,j}$ to $a_{i,j}$ is less than $\frac{1}{i}$

ii) no three of the $b_{i,j}$ are collinear.

This can be done as follows: choose $b_{2,1}$ and $b_{3,1}$ to be any points that is within $\frac{1}{2}$ and $\frac{1}{3}$ of $a_{2,1}$ and $a_{3,1}$, respectively. Now take $b_{3,2}$ to be a point within $\frac{1}{3}$ of $a_{3,2}$ and not lying on the line determined by $b_{2,1}$ and $b_{3,1}$. Inductively, take $b_{i,j}$ to be a point within $\frac{1}{i}$ of $a_{i,j}$ and not lying on any of the lines determined by the points $b_{2,1}, b_{3,1}, b_{3,2}, \ldots, b_{i,1}, b_{i,2}, \ldots, b_{i,i-1}$.

Our claim now is that the set of all $b_{i,j}$ is dense in the square $ABCD$. Consider an arbitrary point $P_0 = (x_0, y_0)$ in the interior of $ABCD$ and let $\epsilon > 0$ be small enough so that $B_\epsilon(P_0)$ is contained in $ABCD$. There are an infinite number of rationals in the interval $(x_0 - \frac{\epsilon}{4}, x_0 + \frac{\epsilon}{4})$ so let r_n be such a rational with $n > \frac{4}{\epsilon}$. The $a_{n,i}$ are within $\frac{1}{n} < \frac{\epsilon}{4}$ of each other, so there exists some $a_{n,k}$ that is within $\frac{\epsilon}{2}$ of P_0. The triangle inequality tells us that since $b_{n,k}$ is within $\frac{\epsilon}{4}$ of $a_{n,k}$ we can conclude $b_{n,k} \in B_\epsilon P_0$, so the $b_{i,j}$ are dense in $ABCD$.

Now take the function f to be the characteristic function on the set of all $b_{n,i}$ and let ξ, η be an arbitrary set of orthogonal directions. For each fixed value of ξ the function f is evaluated at at most two of the $b_{n,i}$, so f is discontinuous on a set of measure zero and constantly zero elsewhere on the given line. Hence

$$\int_{\eta_1}^{\eta_2} f d\xi = 0$$

for every value of η and the integral

$$\int_\alpha^\beta \int_{\eta_1}^{\eta_2} f \, d\xi \, d\eta$$

exists and is identically zero.

On the other hand, the Riemann integral

$$\int\int_{ABCD} f \, dA$$

does not exist since f is discontinuous on a dense set in $ABCD$ when f is no longer restricted to a line.

Q.E.D.

4 Appendix

Fubini's Theorem provides us with sufficient conditions for when plane integration and iterated integration are equivalent when considering Riemann integrals. Below is a statement of this theorem [6].

Fubini's Theorem. Let $R = \{(x, y) \in \mathbf{R^2} : a \leq x \leq b, c \leq y \leq d\}$, and let f be an integrable function on R. Suppose that for each $y \in [c, d]$, the function f_y defined by $f_y(x) = f(x, y)$ is integrable on $[a, b]$, and the function $g(y) = \int_a^b f(x, y)dx$ is integrable on $[c, d]$. Then

$$\int\int_R f dA = \int_c^d \left[\int_a^b f(x, y)dx\right] dy.$$

Remark. Having introduced the notion of a Kakeya set and demonstrating such a set with arbitrarily small area, we are naturally inclined to ask the question: Does there exist a Kakeya set with zero Jordan (or even Lebesgue) measure? As far as I have been able to find this question has neither been answered to the negative nor has a sufficient example been constructed, so I state it here as an open problem.

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