Regularity of split-type Hamiltonians for quilts

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Abstract

Suppose $M = M_0 \times \ldots \times M_N$ is a product symplectic manifold. We show that, under certain hypotheses, there is a Hamiltonian perturbation *of split-type* for which all perturbed pseudoholomorphic curves are regular.

Contents

1 Introduction

Let M be a symplectic manifold, $L_{(0)}, L_{(1)} \subset M$ Lagrangian submanifolds, and J a possibly time-dependent compatible almost complex structure on M. Also fix a Hamiltonian $H \in \mathcal{C}^\infty(I \times M)$, where $I := [0,1]$ is the unit interval, and let X^H denote the associated Hamiltonian vector field. In Floer theory, one is interested in (J,H)-**holomorphic strips**, which are solutions

$$u : (\mathbb{R} \times I, \mathbb{R} \times \{0\}, \mathbb{R} \times \{1\}) \longrightarrow (M, L_{(0)}, L_{(1)})$$

to the perturbed Cauchy-Riemann equation

$$0 = \overline{\partial}_{J,H} u := \partial_s u + J(u)(\partial_t u - X^H(u)).$$

We say that the pair (J,H) is **regular** if the operator $\overline{\partial}_{J,H}$ is transverse to the zero section at all (J,H)-holomorphic strips with finite energy; see [3]. The usefulness of this term is that, when (J,H) is regular, then the set of finite-energy (J,H)-holomorphic strips is a smooth manifold.

It is well-known that, given $M, L_{(0)}, L_{(1)}$ and J, there is a large set of Hamiltonians H for which (J,H) is regular; see [2, Theorem 5.1 (ii)]. In this paper we prove a refinement of this result to the following product case that is relevant to *quilted* Floer theory: Suppose $M_0, \ldots, M_{N-1}$ are symplectic manifolds. To simplify the notation, we discuss only the case where N is even (see [8, Section

4.3] for how to extend the notation to the odd case), however all results hold for N odd as well. Consider the product

$$M := M_0^- \times M_1 \times M_2^- \times M_3 \times \ldots \times M_{N-1};$$

the negative sign in the superscript M_{2j}^- indicates we have replaced the symplectic form on M_{2j} with its negative. Suppose that for each $0 \leq j \leq N-1$ we are equipped with a Lagrangian submanifold $L_{j(j+1)} \subset M_j^- \times M_{j+1}$, where here and below we are taking j modulo N. Consider

$$\left.\begin{array}{lll} L_{(0)} & := & L_{01} \times L_{23} \times \ldots \times L_{(N-2)(N-1)} \\ L_{(1)} & := & \Big(L_{12} \times L_{34} \times \ldots \times L_{(N-1)0}\Big)^T \end{array}\right\} \subset M,$$

where

$$((a_1, a_2), (a_3, a_4), \ldots, (a_{N-1}, a_0))^T = (a_0, a_1, a_2, \ldots, a_{N-1})$$

is the formal shift of coordinate entries. It is straight-forward to check that the $L_{(j)}$ are Lagrangian submanifolds. The relevant almost complex structures and Hamiltonians are those of **split-type**, meaning that they are of the form

$$J = \sum_j (-1)^{j+1} \mathrm{proj}_{M_j}^* J_j, \qquad H = \sum_j (-1)^{j+1} \mathrm{proj}_{M_j}^* H_j,$$

where J_j (resp. H_j) is an almost complex structure (resp. Hamiltonian) on M_j, and $\mathrm{proj}_{M_j} : M \to M_j$ is the projection. When each J_j is compatible as an almost complex structure on M_j, then J is compatible with the structure on M. For convenience, we set

$$\underline{M} := (M_j)_j, \qquad \underline{L} := (L_{j(j+1)})_j, \qquad \underline{J} := (J_j)_j, \qquad \underline{H} := (H_j)_j.$$

Note that specifying $\underline{J}$ (resp. $\underline{H}$) is equivalent to specifying J (resp. H), provided one knows that J (resp. H) is of split-type. Consequently, we will often identify $\underline{J}$ and J (resp. $\underline{H}$ and H).

A Lagrangian $L \subset M_- \times M_+$ in a product symplectic manifold is called **quasisplit** if for every $(x_-, x_+) \in L$, the intersection $(T_{x_-} M_- \times \{0\}) \cap T_{(x_-, x_+)} L$ is independent of x_+ and the intersection $(\{0\} \times T_{x_+} M_+) \cap T_{(x_-, x_+)} L$ is independent of x_-. We say that the tuple $\underline{L}$ is **quasisplit** if this is the case of each $L_{j(j+1)}$.

The main result of the present paper says that, given $\underline{M}$, quasisplit $\underline{L}$, and compatible $\underline{J}$, there is a large set of split-type $\underline{H}$ for which (J, H) is regular. That is, we will show that regularity can be achieved within the class of Hamiltonians that are of split-type. The dual problem in which $\underline{H}$ is fixed, but $\underline{J}$ is perturbed was considered in [8, Theorem 5.2.3] and [9]. However, for certain applications, it is often more natural to perturb the Hamiltonian. For example, this could be the case if the symplectic manifolds are equipped with a Kähler structure, since generic perturbations of the complex structure destroys integrability (of course, there may still be integrable complex structures that give regularity, but statements to this effect require more refined analysis that is likely special to the Kähler manifold).

To state the results precisely, we recall (see [6]) that a (J, H)-holomorphic strip u has finite energy if and only if $u(s, \cdot)$ converges in s (exponentially and uniformly in t) to a map $x : (I, \{0\}, \{1\}) \to (M, L_{(0)}, L_{(1)})$ satisfying

$$\partial_t x = X^H(x).$$

We call such an x an H**-perturbed intersection point**, and we say that x is **non-degenerate** if the linearized operator is transverse at x to the zero section. The terminology is coming from the fact that the set of H-perturbed intersection points can be canonically identified with the set of intersection points of $L_{(0)}$ and the time-1 Hamiltonian flow of $L_{(1)}$ under H; moreover, x is non-degenerate if and only if the associated intersection points is transverse. Let

$$\mathrm{Ham}(\underline{L}) \subseteq \bigoplus_{j=0}^{N-1} \mathcal{C}^\infty(I \times M_j, \mathbb{R})$$

denote the set of split-type Hamiltonian perturbations $\underline{H}$ for which all $\underline{H}$-perturbed generalized intersection points are non-degenerate. It can be shown that $\mathrm{Ham}(\underline{L})$ is open and dense in the space $\oplus_{j=0}^{N-1}\mathcal{C}^\infty(I \times M_j, \mathbb{R})$ of all split-type Hamiltonians; see [8, Proposition 5.2.1].

Proposition 1.1. *Fix $\underline{M}$, quasisplit $\underline{L}$, and (possibly time-dependent) split-type compatible almost complex structure J_{fix}. Then there exists a nonempty subset*

$$\mathrm{Ham}_{\mathrm{reg}}(\underline{L}) := \mathrm{Ham}_{\mathrm{reg}}(\underline{L}, J_{\mathrm{fix}}) \subseteq \mathrm{Ham}(\underline{L})$$

of split type Hamiltonians H such that $H(0, \cdot) = H(1, \cdot) = 0$, and the pair (J_{fix}, H) is regular for Floer theory.

The proof will show that the set $\mathrm{Ham}_{\mathrm{reg}}(\underline{L})$ is actually quite large: it is comeager in the set of all Hamiltonians which, at the perturbed intersection points, agree to second order with a fixed Hamiltonian $H_{\mathrm{fix}} \in \mathrm{Ham}(\underline{L})$. Before proving this result, we make several remarks.

Remark 1.2. *(a) The quasisplit assumption drastically simplifies the proofs of similar results (see the introduction to [9]). It would be interesting to investigate whether the proposition continues to hold without this assumption.*

(b) Our proof will also show that, for any $k \in \{0, \ldots, N-1\}$, there exists a split type Hamiltonian $\underline{H} = (H_j)_{j=0}^{N-1}$ such that (i) $H_j = 0$ for $j \neq k$ and (ii) the linearized Cauchy-Riemann operator is surjective at all $(\underline{J}, \underline{H})$-holomorphic curves $\underline{v} = (v_j)_j$ with $\partial_s v_k \neq 0$.

(c) Any Lagrangian arising from the moduli space of flat connections on a compression body (e.g., as in [7], and [1]) satisfies the quasisplit assumption. This is because each arises as a fibered coisotropic. See [9, Introduction] and [7, Lemma 3.4.5].

2 Proof of Proposition 1.1

The proof we give will follow the proof of [2, Theorem 5.1 (ii)] quite closely, with various modifications coming from [8] and [9]. We recall the details, emphasizing in particular how the argument in [2] can be extended to accommodate our split-type requirements. We note also that in [2], the authors consider a fixed point problem for a symplectomorphism, whereas here we are considering a Lagrangian intersection problem and hence a boundary-value problem. For the most part, this is only a superficial difference since both set-ups define Fredholm problems. However, as pointed out in [5], one does need to take care when considering issues pertaining to 'regular' or 'injective' points at the boundary. In our analysis below we will deal with *interior* regular points, which avoids these more subtle issues.

Fix a split-type Hamiltonian

$$\underline{H}_{\text{fix}} = \left(H_{\text{fix},j}\right)_j \in \text{Ham}(\underline{L})$$

for which all perturbed Lagrangian intersection points are non-degenerate. As usual, write $H_{\text{fix}} = \sum_j (-1)^{j+1} \text{proj}^*_{M_j} H_{\text{fix},j}$. Since $\text{Ham}(\underline{L})$ is open and dense, we may assume $H_{\text{fix},j}(0,\cdot) = H_{\text{fix},j}(1,\cdot) = 0$ vanishes at the boundary, for each j. For $k \in \mathbb{N}^{\geq 2} \cup \{\infty\}$, consider the space

$$\mathcal{C}^k(M, H_{\text{fix}})$$

of Hamiltonians H such that

- H is of split-type: $H = \sum_j (-1)^{j+1} \text{proj}^*_{M_j} H_j$ for some $H_j : I \times M_j \to \mathbb{R}$;
- each H_j (and hence H) is of Hölder class $\mathcal{C}^k$;
- each H_j (and hence H) vanishes at the boundary $H(0,\cdot) = H(1,\cdot) = 0$; and
- H agrees with H_{fix} to second order on the set of H_{fix}-perturbed intersection points.

As usual, set $\underline{H} := \left(H_j\right)_j$. We will show that for all finite $k \geq 2$, the set

$$\left\{ H \in \mathcal{C}^k(M, H_{\text{fix}}) \,|\, \ (J_{\text{fix}}, H) \ \text{ is regular} \right\}$$

is comeager in $\mathcal{C}^k(M, H_{\text{fix}})$. By the Taubes trick [2] [4], this implies the result for $k = \infty$, and so will complete the proof of the proposition.

Note that since each $H \in \mathcal{C}^k(M, H_{\text{fix}})$ agrees with H_{fix} to second order at the intersection points, the sets of H- and H_{fix}-perturbed intersection points are identical,

$$\left\{ x(t) \;\middle|\; \frac{d}{dt} x = X^H(x) \right\} = \left\{ x(t) \;\middle|\; \frac{d}{dt} x = X^{H_{\text{fix}}}(x) \right\}$$

and hence independent of the choice of H (of course, they depend on H_{fix}). Let x^-, x^+ be elements of this set. Denote by $\mathcal{B}$ the space of maps $v : \mathbb{R} \times I \to M$ that are locally of class $W^{1,p}$ ($p > 2$), that have Lagrangian boundary conditions

$$v(s,j) \in L_{(j)}, \qquad j = 0, 1,$$

and that converge to $x^\pm$:

$$\lim_{s\to\pm\infty} v(s,t) = x^\pm(t), \qquad \lim_{s\to\pm\infty} \partial_s v(s,t) = 0$$

uniformly in t. We also require that, for each $v \in \mathcal{B}$, there are trivializations $\Psi^\pm(t) : \mathbb{R}^{2n} \to T_{x^\pm(t)}M$, some large $S > 0$, and functions

$$\xi^- \in W^{1,p}\left((-\infty, -S] \times I, \mathbb{R}^{2n}\right), \qquad \xi^+ \in W^{1,p}\left([S, \infty) \times I, \mathbb{R}^{2n}\right)$$

such that

$$v(s,t) = \exp_{x^\pm(t)}\left(\Psi^\pm(t)\xi^\pm(s,t)\right).$$

Then $\mathcal{B}$ is a smooth Banach manifold with tangent space given by

$$T_v\mathcal{B} = \left\{ \xi \in W^{1,p}(v^*TM) \,\middle|\, \xi(s,j) \in T_{v(s,j)}L_{(j)}, \qquad \lim_{s\to\pm\infty} \xi(s,\cdot) = 0 \right\}.$$

(Since $p > 2$, it follows by Sobolev embedding that the above functions are continuous and so the Lagrangian boundary conditions make sense.) Consider the bundle $\mathcal{E} \to \mathcal{B} \times \mathcal{C}^k(M, H_{\text{fix}})$ whose fiber over (v, H) is the space of L^p vector fields along v:

$$\mathcal{E}_{(v,H)} := L^p(v^*TM).$$

Then the assignment $\mathcal{F} : (v, H) \mapsto \bar{\partial}_{J_{\text{fix}},H} v$ defines a section of $\mathcal{E}$.

At this stage, it suffices to show that $\mathcal{F}$ is transverse to the zero section, and so the universal moduli space $\mathcal{M}(x^-, x^+, J_{\text{fix}})$, consisting of pairs $(v, H) \in \mathcal{B} \times \mathcal{C}^k(M, H_{\text{fix}})$ with $\bar{\partial}_{J_{\text{fix}},H} v = 0$, is a smooth manifold. Once we have shown this, the desired assertion follows from standard arguments by applying the Sard-Smale theorem to the (Fredholm) projection $\mathcal{M}(x^-, x^+, J_{\text{fix}}) \to \mathcal{C}^k(M, H_{\text{fix}})$ sending (v, H) to H; see [3].

Observe that $\mathcal{F}$ is transverse to the zero section if and only if the linearization

$$\begin{aligned} D\mathcal{F}_{(v,H)} : T_v\mathcal{B} \times \mathcal{C}^k(M, H_{\text{fix}}) &\longrightarrow \mathcal{E}_{(v,H)} \\ (\xi, h) &\longmapsto \mathcal{D}_v\xi - J_{\text{fix}} X_t^h(v) = \mathcal{D}_v\xi - \nabla h(t, v) \end{aligned}$$

is onto, where ∇ is the Levi-Civita connection on M determined by J_{fix}, and $\mathcal{D}_v$ is the linearization of $\overline{\partial}_{J_{\text{fix}},H} v = 0$ in the variable v. Since $\mathcal{D}_v$ is Fredholm, it has closed range. In particular, to prove $\mathcal{F}$ is transverse to the zero section, and therefore to prove the proposition, it suffices to prove that $D\mathcal{F}_{(v,H)}$ has dense range. If this is not the case, then there is some non-zero section $\eta \in L^q(v^*TM)$ with $1/q + 1/p = 1$ with

$$\int_{\mathbb{R}\times I} \langle \mathcal{D}_v \xi, \eta \rangle = 0 \qquad \forall \xi \in T_v\mathcal{B}$$

and

$$\int_{\mathbb{R}\times I} dh(t,v)\, \eta \, dsdt = 0 \qquad \forall h \in \mathcal{C}^k(M, H_{\text{fix}}). \tag{1}$$

The first equation implies that η has Lagrangian boundary conditions and satisfies

$$\mathcal{D}_v^* \eta = 0,$$

where $\mathcal{D}_v^*$ is the formal adjoint of $\mathcal{D}_v$. Elliptic regularity for $\mathcal{D}_v^*$ implies that η is of class $\mathcal{C}^k$ (the coefficients of $\mathcal{D}_v^*$ are of class $\mathcal{C}^k$ since H is only of class $\mathcal{C}^k$). We will use the second equation several times to show that $\eta = 0$ vanishes identically. As observed in [9], we need to break the discussion into several cases, depending on the number of constant components of v. This extra degree of complication is not present in the usual Lagrangian intersection theory, and is an artifact of the requirement that the Hamiltonian be of split-type. In the case analysis below we will refer to the components $v_j := \text{proj}_{M_j} v$ of v. The relevant tuple is

$$\underline{v}(s,t) := (v_0(s, 1-t), v_1(s,t), v_2(s, 1-t), v_3(s,t), \ldots, v_{N-1}(s,t)).$$

The $(1-t)$ is introduced to ensure that the jth component of $\underline{v}$ is (J_j, H_j)-holomorphic on M_j, and not $(-J_j, -H_j)$-holomorphic.

First consider the case where v is constant in s. That is, assume $\partial_s v_j = 0$ for all j. Then $\mathcal{D}_v^*$ is an operator of the form $\nabla_s - A$ where A is self-adjoint. Note that since v is constant, the operator A does *not* depend on s. Moreover, A acts on the (s-independent) space of vector fields with Lagrangian boundary conditions in $T_{x^-}L_{(0)}$ and $T_{x^-}L_{(1)}$. We assumed the Hamiltonian $\underline{H}_{\text{fix}}$ was in the space $\text{Ham}(\underline{L})$, and so the generalized perturbed intersection points $x^{\pm}$ are non-degenerate (they are also equal in this case $x^- = x^+$). This implies that A is invertible. It follows from standard Banach space theory that all operators of the form $\partial_s + A : W^{1,2}(\mathbb{R}, W) \to L^2(\mathbb{R}, L)$ are invertible, whenever $A : W \to L$ is invertible, self-adjoint and s-independent. In particular, $0 = \mathcal{D}_v^* \eta = (\partial_s + A)\eta$ implies that $\eta = 0$ and proves the result in the case where v is constant.

Now suppose v is not constant in s, so $\partial_s v_j = 0$ for some (but perhaps not all) j. Analogous to the relationship between $\underline{v}$ and v, we can write η as a tuple $\eta = (\eta_0, \dots, \eta_{N-1})$, where η_j is a section of $v_j^* TM_j$, and then we set

$$\underline{\eta}(s,t) := (\eta_0(s, 1-t), \eta_1(s,t), \dots, N-1(s,t))\,.$$

The idea is to compare these components of η with the corresponding components of $\partial_s v$.

We first treat the non-constant terms. That is, fix a j for which $\partial_s v_j \neq 0$ does not vanish, and we will show that η_j vanishes. We follow the argument in [2] quite closely. Consider the set $R(\eta_j)$ of **interior regular points** of η_j:

$$R(\eta_j) := \left\{ (s,t) \in \mathbb{R} \times (0,1) \;\middle|\; \begin{array}{l} \partial_s \eta_j(s,t) \neq 0, \;\; \eta_j(s,t) \neq 0, \\ \eta_j(s,t) \notin \eta_j(\mathbb{R} \setminus \{s\}, t) \end{array} \right\}.$$

Then the unique continuation result [2, Theorem 4.3] holds for the operator $\mathcal{D}_v^*$ (which is locally a Cauchy-Riemann operator) applied component-wise to show that, if η_j is not identically zero, then $R(\eta_j)$ is open and dense in $\mathbb{R} \times (0,1)$. In particular, to prove η_j vanishes everywhere, it suffices to show that η_j vanishes on some open set in $\mathbb{R} \times (0,1)$. We will show that η_j vanishes on the open set

$$R(v_j) := \left\{ (s,t) \in \mathbb{R} \times (0,1) \;\middle|\; \begin{array}{l} \partial_s v_j(s,t) \neq 0, \;\; v_j(s,t) \neq x_j^{\pm}(t), \\ v_j(s,t) \notin v_j(\mathbb{R} \setminus \{s\}, t) \end{array} \right\}.$$

of **interior regular points** of v_j. Since $\partial_s v_j$ is not identically zero, it follows from [2, Theorem 4.3] that $R(v_j)$ is open and dense in $\mathbb{R} \times (0,1)$. We claim that $\partial_s v_j(s,t)$ and $\eta_j(s,t)$ are linearly dependent for each $(s,t) \in \mathbb{R} \times (0,1)$. If not, then there is some (s_0, t_0) for which $\partial_s v_j(s_0,t_0)$ and $\eta_j(s_0,t_0)$ are linearly independent. We may assume without loss of generality that (s_0,t_0) is contained in $R(v_j)$, since linear independence is an open condition. It follows from the defining properties of $R(v_j)$ that there exists a neighborhood $\widetilde{U}_0 \subset (0,1) \times M_j$ of $(t_0, v(s_0,t_0))$ such that

$$V_0 := \left\{ (s,t) \in \mathbb{R} \times (0,1) \;\middle|\; (t, v_j(s,t)) \in \widetilde{U}_0 \right\}$$

is a small neighborhood of (s_0,t_0). By shrinking these sets if necessary, we may assume $\widetilde{U}_0 = (t_0 - \delta, t_0 + \delta) \times U_0$ is a product for some neighborhood $U_0 \subset M_j$ of $v_j(s_0,t_0)$. Fix a small $\epsilon > 0$ and consider t with $|t - t_0|$ small. Then there is an embedding

$$g_t^j : B_\epsilon(s_0, 0) \to U_0, \qquad (s,\tau) \mapsto \exp_{v_j(s,t)}(\tau \eta_j(s,t)).$$

The key properties are $g_t^j(s,0) = v_j(s,t)$ and $\partial_\tau g_t^j(s,0) = \eta_j(s,t)$. As observed in [2] (see the proof of Theorem 5.1(ii) and Remark 4.4), since g_t^j is an embedding, we can find a Hamiltonian $h_j : I \times M_j \to \mathbb{R}$ which takes the following form on the image of g_t^j:

$$h_j\left(t, g_t^j(s,\tau)\right) = \tau\beta(\tau)\beta(s-s_0)\beta(t-t_0)$$

for some smooth bump function $\beta \geq 0$ supported in a neighborhood of 0, with $\beta(0) = 1$. Differentiate this in τ and set $\tau = 0$ to get

$$dh_j(t, v_j(s,t))\eta_j(s,t) = \beta(s-s_0)\beta(t-t_0).$$

Then h_j pulls back to a Hamiltonian h on M of split-type and clearly satisfies

$$\int_{\mathbb{R}\times I} dh(t, v(s,t))\, \eta(s,t) > 0.$$

This contradicts (1), so η_j and $\partial_s v_j$ are linearly dependent. (Strictly speaking, h is not in $\mathcal{C}^k(M, H_{\text{fix}})$ since it vanishes near the elements of $\mathcal{I}_{\underline{H}_{\text{fix}}}(\underline{L})$, and so does not agree with H_{fix}. However, h can clearly be modified so that this is the case. Note that the above construction builds in the condition that $h(t,\cdot)$ vanishes at $t = 0, 1$.)

Let $C(v_j)$ denote the (discrete) set of points $(s,t) \in \mathbb{R} \times (0,1)$ for which $\partial_s v_j(s,t) \neq 0$. The argument of the previous paragraph shows that we can write

$$\eta_j(s,t) = \lambda(s,t)\partial_s v_j(s,t)$$

for some uniquely determined $\lambda : C(v_j) \to \mathbb{R}$. The next step is to show that $\partial_s \lambda = 0$. If not, then there is some $(s_0, t_0) \in R(v_j)$ with $\partial_s \lambda(s_0, t_0) \neq 0$. Find a function $\rho : \mathbb{R} \times I \to \mathbb{R}$ supported in a neighborhood V_0 of (s_0, t_0) with $\int_{V_0} \rho\, \partial_s \lambda \neq 0$. Integration by parts gives

$$0 \neq \int_{V_0} \partial_s \rho\, \lambda.$$

Now choose $h_j : I \times M_j \to \mathbb{R}$ such that $h_j(t, v_j(s,t)) = \rho(s,t)$ (the existence of h_j follows from the fact that (s_0, t_0) lies in $R(v_j)$, see [2, Remark 4.4]). Then h_j pulls back to a split-type Hamiltonian $h : I \times M \to \mathbb{R}$ on M, and we have

$$dh(t,v)\eta = \lambda \partial_s \rho$$

which again contradicts (1). It follows that $\lambda(s,t) = \lambda(t)$ is independent of s, and so is defined on all of $\mathbb{R} \times (0,1)$ since $C(v_j)$ is discrete. So we have

$$\eta_j(s,t) = \lambda(t)\partial_s v_j(s,t).$$

Since $v_j(s,t)$ satisfies a Cauchy-Riemann equation, it follows that $\eta_j(s,t)$ does as well. So by the Carleman similarity principle [2, Corollary 2.3] it follows that the set of points where $\eta_j(s,t)$ vanishes is discrete, and so $\lambda(t)$ must be non-zero for all t. We may assume $\lambda > 0$, otherwise we replace η_j by $-\eta_j$.

We have $\mathcal{D}_v = \nabla_s + A_s$ and $\mathcal{D}_v^* = \nabla_s - A_s$ for some (s-dependent) operator A_s which is self-adjoint when restricted to forms with Lagrangian boundary conditions. Then by the relations $\mathcal{D}_v \partial_s v = 0$, $\mathcal{D}_v^* \eta = 0$, and the Lagrangian boundary conditions of η and $\partial_s v$, we have

$$\begin{aligned} \frac{d}{ds} \int_0^1 \langle \eta, \partial_s v \rangle \, dt &= \int_0^1 \left(\langle \eta, \nabla_s \partial_s v \rangle + \langle \nabla_s \eta, \partial_s v \rangle \right) \, dt \\ &= \int_0^1 \left(-\langle \eta, A_s \partial_s v \rangle + \langle A_s \eta, \partial_s v \rangle \right) \, dt \\ &= 0 \end{aligned} .$$

So $\int_0^1 \langle \eta, \partial_s v \rangle \, dt$ is independent of s. On the other hand, the previous paragraph applies to all j with $\partial_s v_j$ non-zero, and shows

$$\int_0^1 \langle \eta, \partial_s v \rangle \, dt = \int_0^1 \lambda |\partial_s v|^2 \, dt > 0.$$

But this is independent of s, so integrating over $\mathbb{R}$ gives

$$\int_{\mathbb{R} \times I} \langle \eta, \partial_s v \rangle \, dt = \infty.$$

This contradicts the assumption that $\eta \in L^q$ and $\partial_s v \in L^p$, with $1/p + 1/q = 1$. It follows that $\eta_j = 0$ whenever $\partial_s v_j$ is non-zero. If we are in the extreme case where $\partial_s v_j = 0$ for *all* j, then this already proves the proposition. However, we still need to address the case where $\partial_s v \neq 0$, but there are some values of k for which $\partial_s v_k \equiv 0$. We follow [9], and first note the following observation:

Lemma 2.1. *Fix a complex vector space (V, J), a totally real subspace $\Lambda_0 \subseteq V$, and a path of totally real subspaces $s \mapsto \Lambda_1(s) \subseteq V$. Also fix a translationally-invariant complex connection ∇ on V. Suppose $\eta : \mathbb{R} \times I \to V$ is J-holomorphic*

$$\nabla_s \eta(s,t) + J \nabla_t \eta(s,t) = 0$$

and has totally real boundary conditions

$$\eta(s,0) \in \Lambda_0, \qquad \eta(s,1) \in \Lambda_1(s).$$

Then $\eta \equiv 0$. The same conclusion holds if η is $(-J)$-holomorphic (anti-J-holomorphic).

We prove this lemma at the end. To finish the proof of Proposition 1.1, suppose $\partial_s v_k \equiv 0$, but $\partial_s v_{k-1} \neq 0$. We have just seen that this latter assumption implies that $\eta_{k-1} = 0$. This provides isotropic boundary conditions for η_k; that is, the boundary restriction $\eta_k(\cdot, 0) : \mathbb{R} \to v_k(\cdot, 0)^* TM_k$ takes values in

$$\mathrm{proj}_{T_{y_0} M_k} \left\{ T_{(v_{k-1}(s,1), y_0)} L_{(k-1)k} \cap \left(\{0\} \times T_{y_0} M_k \right) \right\} \subset T_{y_0} M_k \tag{2}$$

and we have set

$$y_0 := v_k(s, 0),$$

which is constant in s by assumption. The key point is that we have assumed the $L_{(k-1)k}$ is quasisplit, which implies that (2) is also independent of s. In any complex trivialization

$$v_k^* TM_k \cong \mathbb{R} \times I \times V,$$

the isotropic space (2) pulls back to a subspace in V contained in a totally real subspace $\Lambda_0 \subset V$. Note that $v_k^* J_k$ is constant in (s,t), since v_k is constant, by assumption. So this almost complex structure pulls back to a constant complex structure on V. This puts us in the realm of Lemma 2.1, and it follows that $\eta_k = 0$ vanishes identically.

Proof of Lemma 2.1. By assumption, η lies in the kernel of the operator $\nabla_s + J\nabla_t$, and $A := J\nabla_t$ is self-adjoint, invertible and independent of s. First consider the case where $\Lambda_1(s) = \Lambda_1$ is constant in s. Then the domain of $J\nabla_t$ is independent of s, and it follows from general theory that $\nabla_s + J\nabla_t$ has trivial kernel, so $\eta = 0$.

We reduce the more general case (i.e., when $\Lambda_1(s)$ depends on s) to the case of the previous paragraph by the reflection principle: Write η in coordinates according to $V = \Lambda_1(s) \oplus J\Lambda_1(s)$:

$$\eta(s,t) = a(s,t)\mu(s) + b(s,t)\nu(s) \in \Lambda_1(s) \oplus J\Lambda_1(s).$$

Then the holomorphic condition for η takes the form

$$\nabla_s(a\mu) + (\partial_t b)J\nu = 0, \qquad \nabla_s(b\nu) + (\partial_t a)J\mu = 0.$$

Define a map

$$\widehat{\eta} : \mathbb{R} \times [0,2] \longrightarrow V$$

by declaring $\widehat{\eta}(s,t) = \eta(s,t)$ for $t \leq 1$, and

$$\widehat{\eta}(s,t) = a(s,1-t)\mu(s) - b(s,1-t)\nu(s) \qquad \text{for } t > 1.$$

Then $\widehat{\eta}$ is $W^{1,\infty}$, by the totally real condition. Moreover, it also satisfies the J-holomorphic curve equation and so is $\mathcal{C}^\infty$. Finally, it has Lagrangian boundary conditions of the form

$$\widehat{\eta}(s,j) \in \Lambda_j$$

for $j = 0, 2$ and so we must have $\widehat{\eta} = 0$ by the previous case.

□

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