The quilted Atiyah-Floer conjecture and the Yang-Mills heat flow

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Notation

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$P \to Y$, an $\mathrm{SO}(3)$-bundle

Assume $w_2(P) \in H^2(Y, \mathbb{Z}_2)$ is in the image of a generator of $H^2(Y, \mathbb{Z})/\mathrm{torsion}$

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Quilted Atiyah-Floer conjecture

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If $\mathcal{M}_{\mathrm{inst}}(g, H)$ and $\mathcal{M}_{\mathrm{symp}}(g, H)$ are cobordant, then the conjecture would follow.

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The proof shows that for suitable g, H, there is a diffeomorphism $\mathcal{M}_{\text{inst}}(g, H) \cong \mathcal{M}_{\text{symp}}(g, H)$.

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- M is the moduli space of flat connections on Σ

This is smooth by the assumptions on the bundle. The metric g induces a complex structure on M.

- $\mathcal{M}_{\mathrm{symp}}(g, H) = \left\{ u : \mathbb{R} \times S^1 \to M \mid \overline{\partial} u = 0 \right\}_1 / \mathbb{R}$

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The instanton equation on $\mathbb{R} \times S^1 \times \Sigma$ is:

$$\begin{aligned} \partial_s A + *\partial_t A &= d_A \Phi + *d_A \Psi \\ F_A &= *(\partial_s \Psi - \partial_t \Phi - [\Psi, \Phi]) \end{aligned}$$

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Dostoglou-Salamon show $*(\partial_s \Psi - \partial_t \Phi - [\Psi, \Phi])$ is small for a suitable metric g.

Then they use an implicit function theorem to map holomorphic curves to instantons.

What makes the general case harder?

If Y is not a mapping cylinder, then strips can likely not be avoided. The symplectic side is therefore a boundary-value problem.

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A different approach

In the general situation, holomorphic strips still lift to connections on $\mathbb{R} \times Y$.

For suitable g, these connections have near-minimal Yang-Mills energy.

Instantons have minimal Yang-Mills energy (by definition).

Idea: Use the Yang-Mills heat flow to flow from (lifted) holomorphic strips to instantons.

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Theorem (D. (2014))

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$$\mathcal{M}_{\mathrm{symp}}(g, H) \to \mathcal{M}_{\mathrm{inst}}(g, H).$$

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When Y is a mapping torus, this gives a new proof of Dostoglou-Salamon's result (+ some more techniques to get surjectivity)

Sketch of proof

Construct metrics g_ϵ that degenerate (in a specific way) as ϵ goes to zero.

Find H so all moduli spaces are cut out transversely for all $\epsilon > 0$.

Show there is a positive energy gap for the (g_ϵ, H)-Yang-Mills functional, and show this is independent of $\epsilon > 0$. (This shows the map in the theorem is well-defined.)

Fix a connection A_0 representing a holomorphic curve and let A_τ be the flow. Show there is a constant C such that

$$\|F^+_{A_\tau}\|_{L^2,\epsilon} \leq C \|d^*_{A_\tau} F_{A_\tau}\|_{L^2(Z)}$$

for all τ and all $0 < \epsilon \leq 1$. (This gives injectivity.)

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- Higher degree Donaldson/quilt invariants?

Thank you for your attention!